

ENBE 415
Example Problems
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Example 1.2.1 Calculate the expected times for men swimming 500 and 600 m in competition.

Solution:

The Riegel equation (1.2.1) will be used:

$$t = ax^b$$

From Table 1.2.1, we find

$$a = 596.2 \text{ sec/km}^b$$

$$b = 1.02977$$

For a distance of 500 m,

$$t = 596.2 (0.5)^{1.02977} = 292 \text{ sec (= 4.87 min)}$$

For a distance of 600 m,

$$t = 596.2 (0.6)^{1.02977} = 352 \text{ sec (= 5.87 min)}$$

The extra 100 m requires an extra minute.

Example 1.3.3.1. At the end of the first minute of heavy exercise, what percentages of the energy requirement come from anaerobic and aerobic metabolic processes?

Solution:

From Table 1.3.1, we see that

aerobic contribution = 65-70%

anaerobic contribution = 30-35%

Example 1.3.3.2. Estimate the oxygen requirement to perform a physical work rate of 225 N·m/sec.

Solution:

From page 11, we find that muscular efficiency is about 20-30%. We'll use 20%. Thus,

$$\text{Physiological work} = \frac{\text{physical work}}{0.2} = \frac{225}{0.2} = 1125 \frac{\text{N} \cdot \text{m}}{\text{sec}}$$

From page 11, we find the energy equivalence of oxygen to be 20,900 kN·m/m³O₂. Thus,

$$\begin{aligned} \dot{V}_{\text{O}_2} &= \left(1125 \frac{\text{N} \cdot \text{m}}{\text{sec}} \right) \left(\frac{\text{m}^3 \text{O}_2}{20,900 \text{kN} \cdot \text{m}} \right) \left(\frac{1 \text{kN}}{1000 \text{N}} \right) \\ &= 5.38 \frac{\text{m}^3 \text{O}_2}{\text{sec}} \quad (= 3.22 \text{ L/min}) \end{aligned}$$

Example 1.3.3.3. Calculate the oxygen deficit incurred when a 40-year-old resting female suddenly begins to work at an external work rate of 92 N·m/sec and continues that work for 20 minutes.

Solution:

1. Refer to Figure 1.3.2. The oxygen deficit is the shaded area at the beginning of the exercise. To obtain that area, the difference between the flat line and the curve must be integrated from time = 0 until time = 20 min. The flat line has a mathematical description of:

$$\Delta \dot{V}_{O_2} = \dot{V}_{O_2} \text{ work} - \dot{V}_{O_2} \text{ rest}$$

The curve can be described by:

$$\dot{V}_{O_2} = \Delta \dot{V}_{O_2} (1 - e^{-t/\tau})$$

The difference between the two lines is:

$$\Delta \dot{V}_{O_2} [1 - (1 - e^{-t/\tau})] = \Delta \dot{V}_{O_2} e^{-t/\tau}$$

Integrating this from $t = 0$ to $t = 1200$ sec,

$$\begin{aligned} \int_0^{1200} \Delta \dot{V}_{O_2} e^{-t/\tau} dt &= \Delta \dot{V}_{O_2} (-\tau) e^{-t/\tau} \Big|_0^{1200} \\ &= \Delta \dot{V}_{O_2} (-\tau) (e^{-1200/\tau} - e^{-0/\tau}) \end{aligned}$$

The time constant value can be found from Table 1.4.1 as 49 sec, although data in Figure 1.3.10 might suggest that τ could be somewhat shorter. We'll use $\tau = 49$ sec.

$$\begin{aligned} \text{Thus, } \Delta \dot{V}_{O_2} (-\tau) (e^{-1200/\tau} - e^{-0/\tau}) \\ &= \Delta \dot{V}_{O_2} (-49) (e^{-1200/49} - e^{-0/49}) \\ &= 49 \Delta \dot{V}_{O_2} \end{aligned}$$

2. Estimate $\Delta \dot{V}_{O_2}$

An external work rate of 92 N·m/sec will have an efficiency of about 20%. The physiological work will thus be about:

$$92 \frac{\text{N} \cdot \text{m}}{\text{sec}} \cdot \frac{1}{0.20} = 460 \frac{\text{N} \cdot \text{m}}{\text{sec}}$$

Required oxygen consumption is thus:

$$\begin{aligned} \dot{V}_{O_2} &= \left(460 \frac{\text{N} \cdot \text{m}}{\text{sec}} \right) \left(\frac{1 \text{ m}^3 \text{O}_2}{20,900 \text{ kN} \cdot \text{m}} \right) \left(\frac{1 \text{ kN}}{1000 \text{ N}} \right) && \text{(p 11)} \\ &= 2.20 \times 10^{-5} \frac{\text{m}^3 \text{O}_2}{\text{sec}} \quad (= 1.32 \text{ L/min}) \end{aligned}$$

This is the value of \dot{V}_{O_2} (work).

\dot{V}_{O_2} (rest) can be found by assuming a physiological work rate of 105 N·m/sec at rest (Table 5.2.21). \dot{V}_{O_2} rest for women is about 0.8 – 0.85 that of men (p 391).

$$\begin{aligned}\dot{V}_{O_2}(\text{rest}) &= (0.8)(105 \frac{\text{N} \cdot \text{m}}{\text{sec}}) (\frac{1}{20,900} \frac{\text{m}^3 \text{O}_2}{\text{kN} \cdot \text{m}}) (\frac{1}{1000} \frac{\text{kN}}{\text{N}}) \\ &= 0.40 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec} \text{ (compare with the value of } 0.42 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec in Figure 1.3.2)}\end{aligned}$$

$$\begin{aligned}\text{Thus, } \Delta \dot{V}_{O_2} &= \dot{V}_{O_2} \text{ work} - \dot{V}_{O_2} \text{ rest} \\ &= 2.20 \times 10^{-5} - 0.40 \times 10^{-5} = 1.80 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec}\end{aligned}$$

3. Estimate Oxygen Deficit:

$$\dot{V}_{O_2}(\text{def}) = 49 \Delta \dot{V}_{O_2} = 88.2 \times 10^{-5} \text{ m}^3 \text{O}_2 \quad (= 0.882 \text{ L})$$

4. Check $\dot{V}_{O_2\text{max}}$:

From p 14, $\dot{V}_{O_2\text{max}}$ for a typical 20-year-old male is $4.2 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec}$.

For a 40-year-old male, there is a decline in $\dot{V}_{O_2\text{max}}$ (p 14). Set up a proportion:

$$\begin{aligned}\dot{V}_{O_2\text{max}}|_{40} &= \dot{V}_{O_2\text{max}}|_{20} [1 - (1.00 - 0.70) \frac{(40 - 20)}{(65 - 20)}] \\ &= 3.64 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec}\end{aligned}$$

A 40-year-old female should have a $\dot{V}_{O_2\text{max}}$ 70% of this.

$$\dot{V}_{O_2\text{max}} = (0.7)(3.64 \times 10^{-5}) = 2.55 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec}$$

Thus, $\dot{V}_{O_2} = 2.2 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec} < 2.55 \times 10^{-5} \text{ m}^3 \text{O}_2/\text{sec} = \dot{V}_{O_2\text{max}}$

This means that the final asymptote to the curve in Figure 1.3.2 will be \dot{V}_{O_2} instead of $\dot{V}_{O_2\text{max}}$.

5. Check Maximum Performance Time

$$\begin{aligned}\text{Endurance time} &= 7200 \left(\frac{\dot{V}_{O_{2\max}}}{\dot{V}_{O_2}} \right) - 7020 && \text{(eqn 1.3.6)} \\ &= 7200 \left(\frac{2.55}{2.20} \right) - 7020 = 1325 \text{ sec} > 1200 \text{ sec}\end{aligned}$$

If maximum endurance time were less than 1200, the work would not have been able to be performed for the specified time, and the time limit on the integral would have had to be changed.

Example 1.3.4.1. Pushing a wheelbarrow requires about 450 W of energy expenditure. Compare endurance times of a 30-year-old woman with a 40-year-old man.

Solution:

Since pushing a wheelbarrow requires 450 W of energy expenditure, about 450 W * 20% = 90 W physical work is done.

Oxygen consumption to push the wheelbarrow is:

$$450 \text{ W} * \frac{1 \text{ mL}}{20.18 \text{ W} \cdot \text{sec}} * \frac{60 \text{ sec}}{\text{min}} = \frac{1 \text{ L}}{1000 \text{ mL}} = 1.34 \text{ L/min}$$

Maximum oxygen consumption for a 30-year-old woman is about 2.00 L/min. Thus,

$$t_{\text{wd}} = 120 \left(\frac{2.00 \text{ L/min}}{1.34 \text{ L/min}} \right) - 117 = 62 \text{ min}$$

For the 40-year-old man, $\dot{V}_{\text{O}_2\text{max}} = 2.60 \text{ L/min}$. Thus,

$$t_{\text{wd}} = 120 \left(\frac{2.60 \text{ L/min}}{1.34 \text{ L/min}} \right) - 117 = 116 \text{ min}$$

In reality, the man would probably weigh more and have a 20% higher oxygen consumption for the same task. Thus,

$$\dot{V}_{\text{O}_2} = (1.2)(1.34 \text{ L/min}) = 1.61 \text{ L/min}$$

$$t_{\text{wd}} = 120 \left(\frac{2.60 \text{ L/min}}{1.61 \text{ L/min}} \right) - 117 = 77 \text{ min}$$

Example 1.3.4.2. Rescue climbing sometimes requires an energy expenditure of 700W and must be sustained for up to 110 min. How fit must the person be to qualify to do this job?

Solution: We will find the required $\dot{V}_{O_2\max}$.

First oxygen consumption is:

$$700 \text{ W} * \frac{1 \text{ mL}}{20.18 \text{ W} \cdot \text{sec}} * \frac{60 \text{ sec}}{\text{min}} * \frac{1 \text{ L}}{1000 \text{ mL}} = 2.08 \text{ L/min}$$

Manipulating the endurance equation:

$$\dot{V}_{O_2\max} = \left(\frac{110 \text{ min} + 117 \text{ min}}{120 \text{ min}} \right) (2.08 \text{ L/min}) = 3.93 \text{ L/min}$$

This is an extremely fit individual. A male in his 20s could perform at this level if he is nearly three standard deviations about the mean.

Example 1.3.6.1. Develop a simple mathematical model to describe oxygen uptake kinetics.

Solution: Upon thinking about this problem, we might decide that it makes sense to us if the rate of change of oxygen consumption would be related to the difference between the actual rate and the required rate. So, a larger difference between the actual rate and the required rate would call for a huge rate of change in oxygen consumption rate. As the actual rate of oxygen consumption approaches the required rate, the rate of change of oxygen consumption would slow down. This process is actually illustrated in the rising portion of Figure 1.3.2, where the oxygen deficit is being accumulated.

This process can be described automatically as:

rate of change of oxygen consumption = k (difference between required rate and actual rate of oxygen consumption)

$$\frac{d}{dt} [\dot{V}_{O_2}] = k [\dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}]$$

where \dot{V}_{O_2} = actual rate of oxygen consumption, $\dot{V}_{O_2}(\text{req})$ = required rate of oxygen consumption, t = time, and k = proportionality constant.

Putting in the initial condition:

$$\dot{V}_{O_2} = \dot{V}_{O_2}(0) \text{ at } t = 0$$

$$C e^{-k(0)} = C = \dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}(0)$$

$$C = \dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}(0)$$

Thus,

$$\dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2} = [\dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}(0)] e^{-kt}$$

Therefore,

$$\dot{V}_{O_2} = \dot{V}_{O_2}(\text{req}) - [\dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}(0)] e^{-kt}$$

Remarks:

We know from Figure 1.3.6 that the required rate of oxygen consumption is related to the power produced, so for any given power value the required rate of oxygen consumption is constant and predetermined.

This is an example of a compartmental problem, and the mathematical expression given above is typical of models for compartmental problems.

Example 1.3.6.2 Mathematically solve the model for oxygen uptake kinetics given in Example 1.3.6.1.

Solution:

The equation developed for that problem was:

$$\frac{d[\dot{V}_{O_2}]}{dt} = k[\dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}]$$

This can be solved in a number of ways, one of which will be illustrated here.

Define a new variable

$$x = \dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}$$

Because $\dot{V}_{O_2}(\text{req})$ is not a function of time,

$$\frac{dx}{dt} = 0 - \frac{d\dot{V}_{O_2}}{dt}$$

So, the equation above becomes

$$-\frac{dx}{dt} = kx$$

$$-\frac{dx}{x} = k dt$$

$$-\int \frac{dx}{x} = \int k dt$$

This is an indeterminate integral that requires a constant of integration

$$-\ln x = kt + C$$

$$x = Ce^{-kt} = \dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}$$

At $t = 0$, $\dot{V}_{O_2} = \dot{V}_{O_2}(0)$, the initial rate of oxygen usage

Thus,

$$x = Ce^{-k(0)} = C = \dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}(0)$$

Hence,

$$Ce^{-kt} = [\dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}(0)] e^{-kt} = \dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}$$

$$\text{or, } \dot{V}_{O_2} = \dot{V}_{O_2}(\text{req}) - [\dot{V}_{O_2}(\text{req}) - \dot{V}_{O_2}(0)] e^{-kt}$$

where $k = 1/\tau$ and $\tau = \text{time constant}$

Remark:

Notice that the equation for \dot{V}_{O_2} varies from $\dot{V}_{O_2}(0)$ at $t = 0$ to $\dot{V}_{O_2}(\text{req})$ at $t = \infty$.

Example 2.3.1 Calculate the Cost of Transport for walking, swimming, and running for an average young man.

Solution:

Walking We could take data appearing in Table 5.2.22 or Table 2.3.1. From Table 2.3.1,

$$\begin{aligned}P_i &= 363 \text{ N}\cdot\text{m}/\text{sec} \\S &= 1.56 \text{ m}/\text{sec} \\W &= 686 \text{ N}\end{aligned}$$

$$CT = \frac{363 \text{ N}\cdot\text{m}/\text{sec}}{(1.56 \text{ m}/\text{sec})(686 \text{ N})} = 0.339$$

Swimming From Table 2.3.1,

$$P_i = 781 \text{ N}\cdot\text{m}/\text{sec}$$

From equation 1.2.2,

$$s = \frac{x^{(1-b)}}{a}$$

From Table 1.2.1,

$$\begin{aligned}a &= 596.2 \text{ sec}/\text{km}^b \\b &= 1.02977\end{aligned}$$

Assuming a distance of 0.4 km,

$$\begin{aligned}s &= \frac{0.4^{-0.02977}}{596.2} = 0.00172 \text{ km}/\text{sec} \\&= 1.72 \text{ m}/\text{sec}\end{aligned}$$

$$CT = \frac{781 \text{ N}\cdot\text{m}/\text{sec}}{(1.72 \text{ m}/\text{sec})(686 \text{ N})} = 0.662$$

Running From Table 2.3.1,

$$P_i = 1353 \text{ N}\cdot\text{m}/\text{sec}$$

From Table 5.2.22,

$$s = 4.47 \text{ m}/\text{sec}$$

$$CT = \frac{1353 \text{ N} \cdot \text{m}/\text{sec}}{(4.47 \text{ m}/\text{sec})(686 \text{ N})} = 0.441$$

Remark: Each of these Costs of Transport is less than the migratory dividing line at 2.0, so this man could be a migrator. The Cost of Transport for swimming is probably higher than the other two exercises because of the viscosity and density of water compared to air.

Example 2.4.1 Calculate the approximate Reynolds number of a running human.

Solution: Reynolds number is calculated from equation 4.2.74 as

$$\text{Re} = \frac{dv\rho}{\mu}$$

Taking the circumferential chest measurement of 100 cm (small man or large woman) as the basis for finding an average diameter, assuming a circular cross-section,

$$\pi d = \text{circumference}$$

$$d = \frac{(100 \text{ cm})(10^{-2} \text{ m/cm})}{\pi} = 0.318 \text{ m}$$

A medium walking speed is about 1.3 m/sec whereas a very fast running speed is about 6.7 m/sec. We can assume something in between, say 3 m/sec.

Surrounding the human is air with a density of about 1.20 kg/m³ and viscosity of 1.81 x 10⁻⁵ kg (cm·sec). Thus,

$$\text{Re} = \frac{(0.318 \text{ m})(3 \text{ m/sec})(1.20 \text{ kg/m}^3)}{1.81 \times 10^{-5} \text{ kg/(m} \cdot \text{sec)}} = 63,000$$

Remark: With a Reynolds number this high, the human will generate turbulence in the surrounding air.

Example 2.6.1 A person works at a rate of 70% of $\dot{V}_{O_{2\max}}$. How long would she be expected to work at this pace? How long a rest period would be needed before she could again work at 70% of $\dot{V}_{O_{2\max}}$?

Solution:

From equation 2.6.1,

$$\begin{aligned}t_{\text{exh}} &= 7200 \left(\frac{\dot{V}_{O_{2\max}}}{\dot{V}_{O_2}} \right) - 7020 \\ &= 7200 \left(\frac{1}{0.70} \right) - 7020 = 3266 \text{ sec} = 54 \text{ min}\end{aligned}$$

From equation 2.6.2,

$$\begin{aligned}t_{\text{rest}} &= 528 \ln \left[\left(\frac{\dot{V}_{O_2}}{\dot{V}_{O_{2\max}}} \right) - 0.5 \right] + 1476 \\ &= 528 \ln (0.70 - 0.50) + 1476 \\ &= 626 \text{ sec} = 10 \text{ min}\end{aligned}$$

Example 3.2.2.1 Calculate the flow resistance of the capillaries.

Solution:

Resistance is pressure drop divided by flow rate (equation 3.2.12). According to Table 3.2.6, the mean pressure in the arterioles, just prior to the capillaries, is $8,000 \text{ N/m}^2$, and the mean pressure downstream in the venules, is 2700 N/m^2 . Thus, pressure drop between arterioles and venules is:

$$\Delta p = 8000 - 2700 = 5300 \text{ N/m}^2$$

Resting cardiac output is normally about $83 \text{ cm}^3/\text{sec}$ ($5\text{L}/\text{min}$) from Table 3.2.4. Thus, the resistance of the capillaries is:

$$R = \frac{\Delta p}{\dot{V}} = \frac{5300 \text{ N/m}^2}{83 \text{ cm}^3 / \text{sec}} \cdot 10^6 \frac{\text{cm}^3}{\text{m}^3} = 64 \times 10^6 \frac{\text{N} \cdot \text{sec}}{\text{m}^5}$$

Remark: The resistance calculated above probably includes a small amount of arteriolar resistance and a small amount of venule resistance. If we had chosen the capillary to venule pressure difference, the pressure drop would have been $4000 - 2700 = 1300 \text{ N/m}^2$, and resistance would have been $16 \times 10^6 \text{ N} \cdot \text{sec}/\text{m}^5$, and if we had chosen instead to use the arteriole to capillary pressure drop the pressure difference would have been $8000 - 4000 = 4000 \text{ N/m}^2$, and resistance would have been $48 \times 10^6 \text{ N} \cdot \text{sec}/\text{m}^5$. The actual capillary resistance is probably between 16×10^6 and $48 \times 10^6 \text{ N} \cdot \text{sec}/\text{m}^5$. This is the resistance of all capillaries in parallel.

Example 3.2.2.2. Calculate the flow resistance of an individual capillary.

Solution:

If we assume laminar flow (a dangerous assumption, but justified in the case of a very small diameter capillary),

$$R = \frac{8L\mu}{\pi r_o^4} \quad (\text{equation 3.2.12})$$

From Table 3.2.2, capillary diameter is about 6×10^{-6} m. We don't know the length, so we can calculate resistance per millimeter of length. Thus, $L = 10^{-3}$ m.

The value of viscosity is influenced by the Fahraeus-Lindqvist effect, so we must calculate apparent viscosity. From equation 3.2.21,

$$\mu = \mu_b \left[1 + 4 \frac{\delta}{r_o} \left(\frac{\mu_b}{\mu_p} - 1 \right) \right]^{-1}$$

where $\delta = 1 \times 10^{-6}$ m (p 86), and $\mu_p = 1.1 - 1.6 \times 10^{-3}$ kg/(m·sec) (p 79). We'll use $\mu_p = 1.3 \times 10^{-3}$ kg/(m·sec).

The value of μ_b can be obtained as 8×10^{-3} kg/(m·sec) from Figure 3.2.7A with 45% hematocrit (p 72) and a shear rate of zero.

Thus,

$$\begin{aligned} \mu &= (8 \times 10^{-3}) \left[1 + 4 \left(\frac{1 \times 10^{-6}}{3 \times 10^{-6}} \right) \left(\frac{8 \times 10^{-3}}{1.3 \times 10^{-3}} - 1 \right) \right]^{-1} \\ &= 1.02 \times 10^{-3} \text{ kg/(m·sec)} \end{aligned}$$

Therefore,

$$\begin{aligned} R &= \frac{8(1 \times 10^{-3} \text{ m})(1.02 \times 10^{-3} \text{ kg/(m·sec)})(1 \text{ N·sec}^2/\text{(m·kg)})}{\pi(3 \times 10^{-6} \text{ m})^4} \\ &= 3.2 \times 10^{16} \frac{\text{N·sec}}{\text{m}^5} \end{aligned}$$

Remark:

Actually, the fact that the apparent viscosity was calculated to be less than the plasma viscosity makes no sense, because the plasma viscosity is the lowest possible

value. The reason this happened is because the assumption $\frac{\delta}{r_o} \ll 1$ is not true in this case.

Apparent viscosity can be calculated by comparing equation 3.2.11 with equation 3.2.18 to obtain:

$$\begin{aligned}\mu &= \mu_p \left[1 - \left(1 - \frac{\delta}{r_o} \right)^4 \left(1 - \frac{\mu_p}{\mu_b} \right) \right]^{-1} \\ &= (1.3 \times 10^{-3}) \left[1 - \left(1 - \frac{1 \times 10^{-6}}{3 \times 10^{-6}} \right)^4 \left(1 - \frac{1.3 \times 10^{-3}}{8 \times 10^{-3}} \right) \right]^{-1} \\ &= 1.56 \times 10^{-3} \text{ kg/(m}\cdot\text{sec)}\end{aligned}$$

and $R = 4.9 \times 10^{16} \text{ N}\cdot\text{sec/m}^5$. Actual capillary resistance will be higher than this because of the twists and bends and the occasional red blood cell being pushed through.

Example 3.2.2.3 Calculate the approximate number of capillaries in parallel in the vasculature.

Solution:

From Example 3.2.2.2 we calculated the resistance of an individual capillary 1mm long to be about $4.9 \times 10^{16} \text{ N sec/m}^5$. From Example 3.2.2.1 we calculated the total resistance of all capillaries in parallel to be about $30 \times 10^6 \text{ N}\cdot\text{sec/m}^5$. If the capillaries are in parallel and approximately of equal resistances, then the total resistance is just the resistance of a typical capillary divided by the number that are in parallel. Thus,

$$\begin{aligned} \text{number of capillaries} &= \frac{\text{resistance of one capillary}}{\text{resistance of all capillaries}} \\ &= \frac{4.9 \times 10^{16} \text{ N}\cdot\text{sec/m}^5}{30 \times 10^6 \text{ N}\cdot\text{sec/m}^5} \\ &= 1.6 \times 10^9 \end{aligned}$$

Example 3.2.2.4 Calculate the approximate Reynolds number for a red blood cell flowing through a capillary.

Solution:

The RBC moves through the capillary, but not as fast as surrounding blood plasma. Thus, we first calculate the velocity of plasma.

From Example 3.2.2.3 we calculated approximately 1.6×10^9 capillaries in parallel. Because the entire cardiac output ($83 \times 10^{-6} \text{ m}^3/\text{sec}$) flows through these capillaries, then the volume flow rate through each capillary is:

$$\dot{V} = \frac{83 \times 10^{-6} \text{ m}^3 / \text{sec}}{1.6 \times 10^9 \text{ capillaries}} = 5.2 \times 10^{-14} \text{ m}^3/(\text{sec cap})$$

Velocity is volume flow rate divided by cross-sectional area, and, with capillary diameter of $6 \times 10^{-6} \text{ m}$ (Table 3.2.2):

$$A = \frac{\pi d^2}{4} = \frac{\pi(6 \times 10^{-6} \text{ m})^2}{4} = 2.83 \times 10^{-11} \text{ m}^2/\text{cap}$$

$$v = \frac{\dot{V}}{A} = \frac{5.1 \times 10^{-14} \text{ m}^3/\text{sec}}{2.83 \times 10^{-11} \text{ m}^2} = 1.83 \times 10^{-3} \text{ m/sec}$$

If the plasma travels three times faster than the RBCs, then

$$v_{\text{RBC}} = \frac{1.83 \times 10^{-3}}{3} \text{ m/sec} = 6.12 \times 10^{-4} \text{ m/sec}$$

and the *relative* velocity of RBCs in the plasma stream is:

$$v = 1.83 \times 10^{-3} \text{ m/sec} - 6.12 \times 10^{-4} \text{ m/sec} = 1.22 \times 10^{-3} \text{ m/sec}$$

The RBC has an elliptic cross section, $7.5 \mu\text{m} \times 0.3 \mu\text{m}$. In order to obtain an average diameter needed for the Reynolds number calculation, we first calculate the area of the ellipse and then calculate what circular diameter would provide the same area. This is the diameter we'll use in the Reynolds number calculation.

$$\text{Area of ellipse} = \pi ab = \pi(7.5 \times 10^{-6} \text{ m})(0.3 \times 10^{-6} \text{ m}) = 7.07 \times 10^{-12} \text{ m}^2$$

$$d = \sqrt{\frac{4 \cdot \text{elliptic area}}{\pi}} = 3 \times 10^{-6} \text{ m}$$

The density of plasma is about 1020 kg/m^3 , and the viscosity of about $1.4 \times 10^{-3} \text{ kg/(m}\cdot\text{sec)}$. Thus, the Reynolds number is

$$\begin{aligned} \text{Re} &= \frac{dv\rho}{\mu} = \frac{(3 \times 10^{-6} \text{ m})(1.22 \times 10^{-3} \text{ m/sec})(1020 \text{ kg/m}^3)}{1.4 \times 10^{-3} \text{ kg/(m}\cdot\text{sec)}} \\ &= 2.66 \times 10^{-3} \end{aligned}$$

Remark: Note that one dimension of the RBC ($7.5\mu\text{m}$) is larger than the capillary diameter ($6\mu\text{m}$). RBCs cannot move through the capillary without folding, and it is the friction caused by rubbing against the capillary wall that slows the RBC relative to the plasma.

Example 3.2.3.1 What is the expected heartrate for a 35-year-old man performing work at 70% of his $\dot{V}_{O_2\max}$?

Solution:

$$\text{Resting } \dot{V}_{O_2} = 10\% \dot{V}_{O_2\max} .$$

$$\text{Max hr} = 220 - \text{age} = 220 - 35 = 185 \text{ beats/min}$$

$$\text{predicted hr} = 70 + (0.7 - 0.1)(185 - 70)$$

$$= 139 \text{ beats/min}$$

Example 3.3.1.1 Estimate Parameter Values for the Carotid Sinus Stretch Receptor Equation 3.3.2.

Solution:

Values for the parameters β_+ , β_- , β_0 , and p_{th} can be estimated from Figure 3.3.2, Table 3.2.6, and some simple assumptions. From Figure 3.3.2, we see that pulsatile flow results in a higher frequency output than does a constant, nonpulsatile flow. In pulsatile flow the terms dp/dt must assume both positive and negative values. The first assumption to be made, therefore, is:

Assumption 1: both positive and negative dp/dt result in an increase of receptor output frequency.

Another assumption relates to the relative durations and magnitudes of increasing and decreasing phases of pulsatile pressure:

Assumption 2: both positive and negative dp/dt will have equal magnitudes and durations.

To estimate values for dp/dt , we obtain from Table 3.2.6, that blood pressure in a large artery is $16,700 \text{ N/m}^2$ (systolic) and $10,300 \text{ N/m}^2$ (diastolic).

Assumption 3: the pulsatile pressure in Figure 3.3.2 is due to normal pressure pulses in a large artery.

From page 93, we find that average heart rate is 1.17 beats/sec. Thus, the average beat-to-beat period = $1/1.17 = 0.855 \text{ sec/beat}$.

$$\text{Average } \frac{dp}{dt} \approx \frac{\Delta p}{\Delta t}$$

From assumption 2, $\Delta t =$ one half of the period, or 0.427 sec.

$$\text{Thus, } \frac{\Delta p}{\Delta t} = \frac{16,700 - 10,300}{0.427} = 14,976 \frac{\text{N/m}^2}{\text{sec}}$$

Because we have no better information, we make an additional assumption:

$$\text{Assumption 4: } \beta_- = -\beta_+$$

The negative sign is included to make the increment in discharge frequency positive for a negative dp/dt .

From Figure 3.3.2, the additional discharge frequency due to pulsatile pressure is:

$$\Delta f \approx 25 \text{ pulses/sec}$$

$$\text{Thus, } \Delta f = 2 \beta_+ \left(\frac{\Delta p}{\Delta t} \right)$$

$$\beta_+ = \frac{25}{(2)(14976)} = 8.35 \times 10^{-4} \frac{\text{impulses} \cdot \text{m}^2}{\text{N}}$$

$$\beta_- = -\beta_+ = -8.35 \times 10^{-4} \frac{\text{impulses} \cdot \text{m}^2}{\text{N}}$$

To obtain a value for β_0 and p_{th} ,

Assumption 5: p_{th} is the lowest pressure appearing in the curve in Figure 3.3.2. The validity of this assumption is not too important, because a different value of p_{th} will just result in a different value of β_0 to give the same frequency, f .

$$p_{th} = 7000 \text{ N/m}^2$$

Assumption 6: the value of the constant pressure in Figure 3.3.2 is the average of systolic and diastolic pressures.

$$\text{Thus, } \bar{p} = \frac{16,700 + 10,300}{2} = 13,500 \text{ N/m}^2$$

From Figure 3.3.2,

$$f \approx 90 \text{ at } 13,500 \text{ N/m}^2$$

Assumption 7: the relationship between pressure and firing rate is linear.

Therefore,

$$f = \beta_0 (p - p_{th}) = 90 = \beta_0 (13,500 - 7000)$$

$$\beta_0 = \frac{90}{6,500} = 1.38 \times 10^{-2} \frac{\text{impulses} \cdot \text{m}^2}{\text{sec} \cdot \text{N}}$$

Hence, equation 3.3.2 is:

$$f = (8.35 \times 10^{-4}) \frac{dp}{dt} + 1.38 \times 10^{-2} (p - 7000) \quad \begin{array}{l} \frac{dp}{dt} > 0 \\ p > p_{th} \end{array}$$

$$f = -8.35 \times 10^{-4} \frac{dp}{dt} + 1.38 \times 10^{-2} (p - 7000) \quad \frac{dp}{dt} < 0$$

$p > p_{th}$

or,

$$f = 8.35 \times 10^{-4} \left| \frac{dp}{dt} \right| + 1.38 \times 10^{-2} (p - 7000) \quad p > p_{th}$$

or,

$$f = 8.35 \times 10^{-4} \left[\operatorname{sgn} \left(\frac{dp}{dt} \right) \right] \frac{dp}{dt} + 1.38 \times 10^{-2} (p - 7000) \quad p > p_{th}$$

$$\begin{aligned} \text{where } \operatorname{sgn} \left(\frac{dp}{dt} \right) &= + \text{ for } dp/dt > 0 \\ &= - \text{ for } dp/dt < 0 \end{aligned}$$

Example 3.5.1.1 Force-Length relationship for Hill’s muscle model. Tabulate values for muscle force and relative length for a cat papillary muscle fiber using Hill’s model undergoing static contraction.

Solution:

Refer to the model on the left in Figure 3.4.1, and to force-length diagrams in Figure 3.5.5 and 3.5.6. To illustrate the method, choose a relative length of 1.30 for the contractile element and parallel elastic element (both have the same length). From Figure 3.5.6,

The maximum contractile force is:

$$F_{c \max} = 81 \times 10^{-3} \text{ N}$$

and the parallel element force is

$$F_p = 33 \times 10^{-3} \text{ N}$$

Maximum total force produced by both elements is (from equation 3.5.16):

$$F_{\max} = (1 + \phi)F_{c \max} + F_p$$

where $\phi = 0.02$ (p 136)

$$F_{\max} = (1.02)(81 \times 10^{-3}) + 33 \times 10^{-3} = 115.62 \times 10^{-3} \text{ N}$$

Because the series elastic element must transmit the maximum total force, the force on the series elastic element is $115.62 \times 10^{-3} \text{ N}$. From Figure 3.5.5, this corresponds to a relative length of 0.156.

From equation 3.5.13,

$$L = L_s + L_c = 0.156 + 1.30 = 1.46$$

Maximum force-lengths for the muscle are:

L_c	$F_{c \max}$	F_p	F_{\max}	L_s	L
1.00	0	0	0	0.00	1.00
1.05	19	0	19.38	0.02	1.15
1.10	39	0	39.78	0.12	1.22
1.15	62	0	63.24	0.14	1.29
1.20	86	0	87.72	0.15	1.35
1.25	95	4	98.80	0.15	1.40
1.30	81	33	115.62	0.62	1.46

Values for minimum force can be found using equation 3.5.15 with $\phi = 0.02$. Tabulated values are:

L_c	F_{\min}	L_s	L
1.00	0	0.00	1.00
1.05	19.38	0.00	1.05
1.10	39.78	0.02	1.12
1.15	63.24	0.03	1.18
1.20	87.72	0.04	1.24
1.25	96.90	0.07	1.32
1.30	82.62	0.12	1.42

Example A3.1.1 Approximate Equation 3.3.2 as a finite difference equation:

Solution:

First, let us use the third form of Equation 3.3.2 appearing as a solution in Example 3.3.1.1:

$$f = 8.35 \times 10^{-4} \left[\text{sgn} \left(\frac{dp}{dt} \right) \right] \frac{dp}{dt} + 1.38 \times 10^{-2} (p - 7000) \quad p > p_{th}$$

Using the central difference approximation to the first derivative,

$$\frac{dp}{dt} \approx \frac{p_{i+1} - p_{i-1}}{t_{i+1} - t_{i-1}}$$

So,

$$f = 8.35 \times 10^{-4} \left[\frac{p_{i+1} - p_{i-1}}{t_{i+1} - t_{i-1}} \right] + 1.38 \times 10^{-2} [p_i - 7000]$$

If the time step, $t_{i+1} - t_i$, is designated as Δt ,

$$f = 8.35 \times 10^{-4} \left[\frac{p_{i+1} - p_{i-1}}{2\Delta t} \right] + 1.38 \times 10^{-2} [p_i - 7000]$$

Solving for p_{i+1} ,

$$p_{i+1} = p_{i-1} + 2\Delta t \left[\frac{f - 1.38 \times 10^{-2} (p_i - 7000)}{8.35 \times 10^{-4}} \right] \quad (1)$$

This form of the equation can be used to estimate the next pressure (p_{i+1}) once the frequency, the former pressure (p_{i-1}), and the present pressure (p_i) are known. To start the numerical process, the forward difference approximation is used:

$$\frac{dp}{dt} \approx \frac{p_2 - p_1}{\Delta t}$$

$$f = 8.35 \times 10^{-4} \left[\frac{p_2 - p_1}{\Delta t} \right] + 1.38 \times 10^{-2} [p_1 - 7000]$$

$$p_2 = p_1 + \Delta t \left[\frac{f - 1.38 \times 10^{-2} (p_1 - 7000)}{8.35 \times 10^{-4}} \right] \quad (2)$$

This equation can be used to obtain the value of p_2 from p_1 and f . Once p_1 and p_2 are known, then

$$p_3 = p_1 + 2\Delta t \left[\frac{f - 1.38 \times 10^{-2}(p_2 - 7000)}{8.35 \times 10^{-4}} \right]$$

from the previous Equation (1). Thereafter, Equation (1) can be used to find subsequent values of pressure.

Note: This example is a somewhat backwards illustration. Usually, an equation such as Equation 3.3.2 would be used to find discharge frequency from pressure. In this example, we have found pressure from frequency. By this means, the method was demonstrated, although this would not be a normal way of using Equation 3.3.2.

Example 4.2.2.1 At what rate is CO₂ being added to the atmosphere by the world's population?

Solution:

Assume	1/3 sleeping	75W	energy expenditure
	1/3 resting	125W	
	1/3 working	<u>200W</u>	
		400W	total

$$\dot{V}_{O_2} = 400W * \frac{1 \text{ mL/sec}}{20.18} * \frac{60 \text{ min}}{\text{sec}} * \frac{1 \text{ L}}{1000 \text{ mL}} = 1.2 \text{ L/min}$$

If Respiratory Exchange Ratio \approx 0.9,

$$\dot{V}_{CO_2} = 1.2 \text{ L/min} * 0.9 = 1.1 \text{ L/min}$$

For 5 billion people, total \dot{V}_{CO_2} is 5.4×10^9 L/min. Of course other human activities add a great deal more.

Example 4.3.2.1 What is the maximum resistance of an external breathing device to be unnoticeable?

Solution:

If $\frac{R_{\text{external}}}{R_{\text{internal}}} < 25\%$ for no effect,

and $R_{\text{internal}} = 4.00 \text{ cm H}_2\text{O}\cdot\text{sec/L}$

the $R_{\text{external}} = 4.00 * 0.25 = 1.00 \text{ cm H}_2\text{O}\cdot\text{sec/L}$

Example 4.3.4.1 How much air needs to be stored in a SCUBA tank to sustain an average-fitness swimmer for 30 minutes?

Solution:

From the energy expenditure Table 5.2.22, swimming requires about 800W energy expenditure.

$$\dot{V}_{O_2} = 800W * \frac{1 \text{ mL/sec}}{20.18W} * \frac{60 \text{ sec}}{\text{min}} * \frac{1 \text{ L}}{1000 \text{ mL}} = 2.4 \text{ L/min}$$

From Figure 4.3.26, the corresponding pulmonary ventilation is 60 L/min.

Thus, the total amount of air = 60 L/min * 30 min = 1800 L

Note: We may be tempted to say that a tank containing 2.4 L/min * 30 min = 72 L of pure oxygen is sufficient to sustain the swimmer. That is not true! Because respiratory control depends almost exclusively on CO₂ produced, the swimmer would still need about 1800 L of oxygen in the tank.

Example 5.2.6.1 In order to perform a transplant operation, the patient's body temperature must be reduced to 30°C. This is to be done by routing the blood through a chiller. Assume that the blood can be cooled at most to 25°C. How long will this take for a 70 kg patient?

Solution:

Normal body temperature is 37°C. However, we'll assume that cooling starts when the patient's body temperature is 34.7°C.

A resting body pumps blood at 5L/min. This is about 5 kg/min.

If we assume that blood enters the body at 25°C and leaves at body temperature, the heat balance is:

$$\begin{aligned} \text{rate of heat in} - \text{rate of heat out} + \text{rate of heat generation} \\ = \text{rate of change of heat stored} \end{aligned}$$

The rates of heat in and out will depend on convection in the arteries and veins, and is equivalent to the change of heat storage in the blood as it flows through the chiller:

$$(\text{rate of heat in} - \text{rate of heat out}) = - \dot{m}_{\text{blood}} c_p (\theta_{\text{blood}} - 25)$$

and θ_{blood} soon equilibrates with body temperature.

The rate of heat generation is fairly small for an anaesthetized patient, and it decreases as the patient cools. Thus, the rate of heat generation will be assumed negligible for this example. Otherwise, it could be estimated from the BMR with temperature dependence.

The rate of change of heat stored in the body is:

$$m_{\text{body}} c_p \frac{\Delta\theta}{\Delta t}$$

Thus, the approximate heat balance becomes

$$m_{\text{body}} c_p \frac{\Delta\theta}{\Delta t} = - \dot{m}_{\text{blood}} c_p (\theta - 25)$$

If the specific heat of blood and body are nearly the same,

$$\text{then, } m_{\text{body}} \frac{\Delta\theta}{\Delta t} = - \dot{m}_{\text{blood}} (\theta - 25)$$

Using the forward difference approximation to the derivative (Appendix 3.1),

$$m_{\text{body}} \frac{(\theta_{i+1} - \theta_i)}{\Delta t} = -\dot{m}_{\text{blood}} (\theta_i - 25)$$

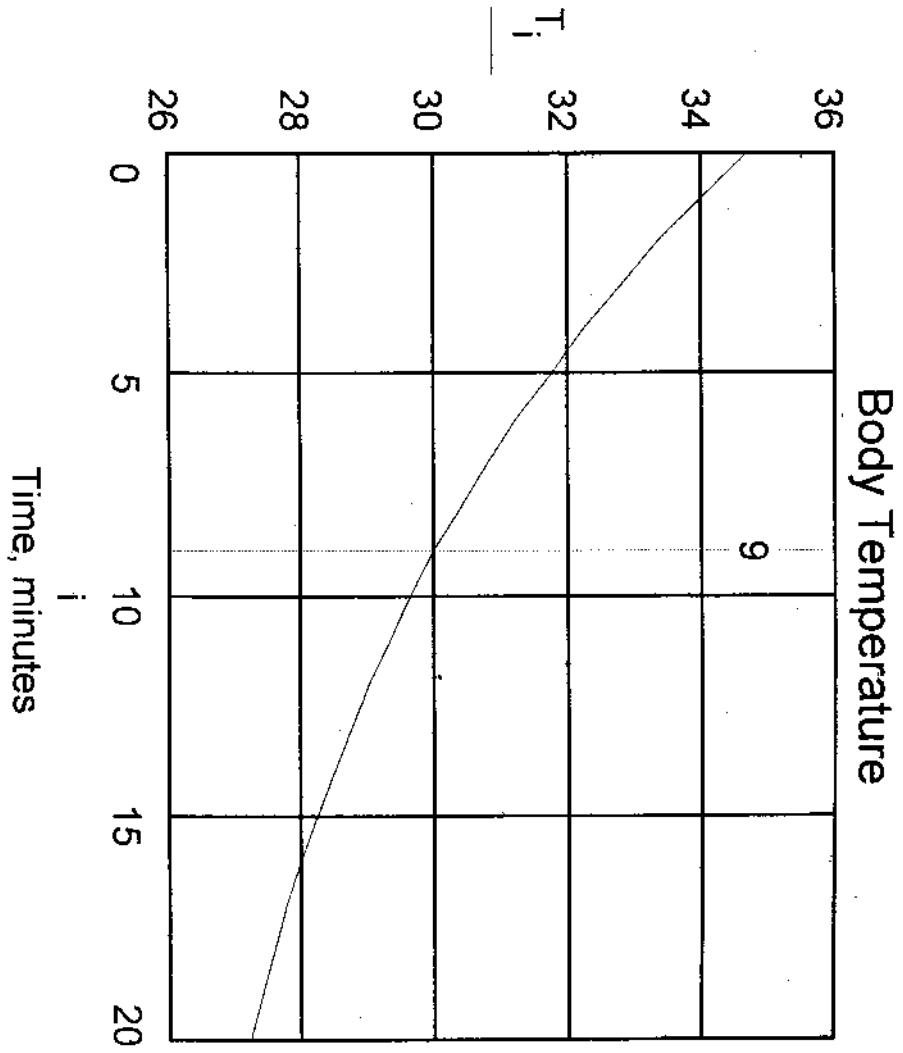
Solving for θ_{i+1} ,

$$\theta_{i+1} = \theta_i - \frac{\dot{m}_{\text{blood}} \Delta t (\theta_i - 25)}{m_{\text{body}}}$$

Using $m_{\text{body}} = 70 \text{ kg}$
 $\dot{m}_{\text{blood}} = 5 \text{ kg/min}$
 $\Delta t = 1 \text{ min}$
 $\theta_1 = 34.7^\circ\text{C}$

We obtain the following graph, from which we see that cooling will occur in 9 minutes.

Body Temperature deg C



Example 5.5.1.1 What would be the rate of moisture that would be expected to be added to the air in a room by a person working at a rate of 150W physical work and who is in thermal equilibrium in the room at 105W physiological work?

Solution:

The person working at 150W physical work probably has a muscular efficiency of 15–20%. Thus, the physiological work is:

$$\text{Physiol work rate} = 150\text{W}/0.20 = 750\text{W}$$

If the person is in thermal equilibrium at 105W, then the excess heat to be removed by sweating is about

$$E_{\text{req}} \approx 750\text{W} - 105\text{W} = 645\text{W}$$

The rate of sweat production is

$$\begin{aligned} \dot{m}_{\text{sw}} &= 7.75 \times 10^{-6} E_{\text{req}} (E_{\text{max}}/A)^{-0.455} \\ &= 7.75 \times 10^{-6} (645)(1200/1.8)^{-0.455} \\ &= 2.59 \times 10^{-4} \text{ kg/sec} \end{aligned}$$